

ON MEANING AND MEASURING: A PHILOSOPHICAL AND HISTORICAL VIEW

ENRIC TRILLAS & RUDOLF SEISING
European Centre for Soft Computing, Mieres, Asturias

Resumen

El objetivo de este trabajo es reconsiderar el punto de vista defendido por los pensadores que se pueden adscribir al famoso ‘Círculo de Viena’ sobre el significado y, especialmente, sobre los enunciados asignificativos. Para ello, procedemos usando la típica forma científica, heredera de la tradición del Círculo, de medir empíricamente los conceptos adquiridos. Además, proporcionamos algunas notas históricas, así como algunas reflexiones sobre el pensamiento de Karl Menger sobre la carencia de una geometría adecuada para el micromundo.

Palabras clave: Filosofía del Círculo de Viena, asignificatividad, metafísica, medible.

Abstract

This paper just tries to revisit the view on meaning, and particularly in meaningless statements, hold by the thinkers inscribed in the famous “Vienna Circle”. It is done under the typically scientific form of measuring empirically acquired concepts that can be considered in the same tradition of the Circle. In addition, it also contains some historical notes, as well as some related reflexions on Karl Menger’s thought on the lack of a suitable geometry for the microworld.

Keywords: Vienna Circle’s Philosophy, meaningless, metaphysical, measurability.

1. Introduction

1.1. It is well known that, under the influence of Mach’s epistemology (Nagel, 1956) and Wittgenstein’s ‘Tractatus’ (Wittgenstein, 1922), the

Recibido: 13/08/2014. Aceptado: 15/12/2014.

philosophical work of the Vienna Circle, newly founded in 1928 as the Ernst Mach Society (“Verein Ernst Mach”), did mainly turn around the concept of the meaning of propositions, basically understood throughout the possibility of testing them against the ‘reality’ they refer to. It is also well known that, almost unanimously its members refused the meaningless statements they called ‘metaphysical’ that, indeed, tried to expulse from philosophical consideration by arguing that they were something not related with experience and just deserving a careful linguistic analysis under which such meaningless statements would disappear from the philosophical discourse. The Circle’s manifesto “The Scientific Conception of the World” refers to Ernst Mach who “was especially intent on cleaning empirical science, and in the first place, physics, of metaphysical notions. We recall his critique of absolute space which made him a forerunner of Einstein, his struggle against the metaphysics of the thing - in itself and of the concept of substance, and his investigations of the construction of scientific concepts from ultimate elements, namely sense data. In some points the development of science has not vindicated his views, for instance in his opposition to atomic theory and in his expectation that physics would be advanced through the physiology of the senses. The essential points of his conception however were of positive use in the further development of science.” (Verein Ernst Mach, 1929).

It is also worth remembering that most, if not all of the members of the ‘Circle’, actually were not professional philosophers, but professors and researchers of some scientific or experimental oriented disciplines and, hence, coming from the scientific tradition that, starting with Galileo, did found modern science on numerical measures, mathematical models and their continuous checking with reality. Additionally, the scientists working at the ‘Austrian Institute for the Economic Conjuncture’s Research’,¹ like Ludwig von Mises, Oskar Morgenstern and Abraham Wald influenced the Circle’s thinking (Menger, 1994). Actually, the intellectual atmosphere of Vienna was truly impressive in that time (Kraft, 1951).

Concerning the two influences just cited at the beginning, if Ernst Mach was a physicist, Ludwig Wittgenstein was initially educated as an engineer and returned, later on and in his posthumous ‘Philosophical Investigations’ (Wittgenstein, 1958), to look at language from a kind of ‘But, how does it work?’ (Mamdani, Trillas, 2012), a typically engineering point of view.

¹ This institute, in German “Österreichisches Institut für Wirtschaftsforschung” (WIFO), was founded in 1927 under the name “Österreichisches Institut für Konjunkturforschung” by Friedrich August von Hayek und Ludwig von Mises.

Concerning the neighborhood of the Circle and for instance, the famous geometer Karl Menger who, even attending many of its sessions, did not properly belong to the Circle, not only influenced it with his work, but then was influenced by the Circle and the intellectual climate created by it in Vienna and in, at least, for what concerns his affection for what he called “exact thinking” (Menger, 1974) in non-purely mathematic subjects. Moreover, Menger organized his “Mathematical Colloquium” in parallel to the Vienna Circle in 1928 with students and foreign guests, and he published the “Reports of a Mathematical Colloquium” (1931–1937).² All that for saying nothing on the perhaps final time influence, due to the work of Kurt Gödel, who was Menger’s assistant (Menger, 1994).

With these intellectual influences, the Circle tried to introduce in Philosophy the successful forms of reasoning supporting exact science, and was hardly, and sometimes harshly, answered by some professional philosophers. The Circle’s goal was a renewal and modernization of the philosophical thought by approaching it to science and by banishing all that were not with the ‘feet on the Earth’ as per organization of lectures and their publication by Hans Hahn, Karl Menger, Hans Thirring and Hermann Mark, e.g.:

- Crisis and Reconstruction in the Exact Sciences, 1933.³
- Old Problems - New Solutions in the Exact Sciences, 1934.⁴
- Newer Advancements in the Exact Sciences, 1936.⁵

In that time, Rudolf Carnap, another member of the Circle had already immigrated to the US, where he published a book on the rejection of metaphysics. He called “metaphysical all those propositions which claim to represent knowledge about something which is over or beyond all experience, e.g. about the real Essence of things, about Things in themselves, the Absolute, and such like. [...] The sort of propositions I wish to denote as metaphysical may most easily be made clear by some examples: “The Essence and the Principle of the world is Water”, said Thales; “Fire”, said Heraclitus; “the Infinite”, said Anaximander; “Number”, said Pythagoras. “All things are nothing but shadows of eternal ideas which themselves are in spaceless and timeless sphere”, is a doctrine of Plato. From the Monists we learn: “There is only one principle on which all that is, is founded”; but

² The Reports were edited and published by Menger (with the help of Kurt Gödel, Georg Nöbeling, Abraham Wald and Franz Alt) in the *Ergebnisse eines Mathematischen Kolloquiums*. (Dierker, Sigmund 1989).

³ Original in German: *Krise und Neuaufbau in den exakten Wissenschaften*.

⁴ Original in German: *Alte Probleme – Neue Lösungen in den exakten Wissenschaften*.

⁵ Original in German: *Neuere Fortschritte in den exakten Wissenschaften*.

the Dualists tell us: “There are two principles”. The Materialists say: “All that is, is in its essence material”, but the Spiritualists say: “All that is, is spiritual”. To metaphysics (in our sense of the word) belong the principal doctrines of Spinoza, Schelling, Hegel, and - to give at least one name of present time - Bergson.” (Carnap, 1935, 15).

Anyway, the Circle never fully clarified on which scientific-like ground the statements they called metaphysical were identified with those that are meaningless, and why these statements should not be considered in thinking. That they cannot be immediately confronted with some ‘reality’ seems to be, in itself, and by looking at how human thinking is sometimes guided by foggy lights, a little bit metaphysical and even if, at the end, the road towards positivistic thinking opened by Galileo did humiliate the former kind of ‘abstract’ thinking supported by theologians. Concerning the usefulness of meaningless statements for a long run thinking towards something currently unknown, and still parodying Galileo, it could be said ‘Eppur si muove!’.

1.2. This short paper tries, limited to predicates but in the same intellectual tradition of the Circle, to look at how science represents its concepts, often arising from empirical observation and once they are solidified by controlled experimentation, by means of either numerical or qualitative quantities ^[11] helping to transform them in scientifically fertile concepts. At the end, when trying to face an old question, and meaning is such, it is sometimes a good strategy that of restarting to look at it from a new, or at least different, point of view; also, and in some occasions, the question results dissolved into the new perspective.

As it will be seen, if the Circle was mostly influenced by the science of its time, the authors of this paper are mainly influenced by questions arising from the field of fuzzy logic. What is presented could allow concluding that the concept of the meaning of a scientifically-fertile predicate lies in its measurability’s character. Along the argumentation it will be tried to make clear which is the difference between being just meaningless and being a metaphysical predicate.

2. How do people reason on something?

2.1. Said in a plain language, reasoning deals with subjects on which some amount of suitable knowledge is available to those that reason. In the first place and for clarifying their own understanding of the subject’s knowledge components, those who reason should try to introduce some

kind of order, or hierarchy, between the linguistic statements translating the knowledge. For instance, if X is a set of elements x, y, z, \dots , and P is a predicate (a name given to a property p shown by the elements of X), to capture what reflect the elemental, or atomic, statements “ x is P ”, it appears to be necessary to establish some ordering or hierarchy among the elements in X and with respect to how they verify p .

It should be pointed out that most of the predicates to which this writing refers to, cannot be ‘defined’ by an equivalent linguistic expression as it happens, for instance, with the predicate ‘transcendent’ in $X = \mathbf{R}$, the real line. In such cases, we can go back up to Peano’s axioms, that is, that all the information necessary to capture completely the meaning of the predicate is, at least, potentially available. Nevertheless and often, those predicates we will refer to are only ‘describable’ by means of some rules concerning its basic application to the elements in X ; a set of rules not always sufficient to represent the predicate by a subset of X in the way of the Cantor-Zermelo specification axiom. A set of rules in which it is not possible to retrocede, like in the case of “transcendent”, for capturing the meaning of the linguistic terms appearing in their antecedents and consequents some of which, at its turn, only can be described by some other rules of use. Hence, in many times some additional information should be added to better capture, or to refine, the predicate’s use.

Examples

a) Provided some statements on the tallness of the Londoners intervene in a discourse, the first that will be needed is to agree on which reality has in London a statement like “John is tall”, something that requires to simultaneously capture which is the reality of “Peter is short”, both John and Peter being Londoners. Of course, the knowledge on the behavior of the predicate “tall” in London will be not sufficient enough if it is not possible to recognize which one of John and Peter is either taller, or shorter, than the other. Indeed, often people learn the use of names P by bipolarity, by comparing instances of what is P with instances of what is in a family of elemental statements related to P , like it is ‘antonym of P ’, and, in general by how P varies when applied to the elements of X . There should be simultaneity, for instance, in knowing “John is tall” and “Peter is short”. There is no possibility of learning the meaning of an isolated name.

b) It happens analogously when predicating “vague” in a universe of statements X , where to know that “statement x is vague” will be, at least, insufficient up to when it can be stated which of two statements is less, or more, vague than the other. It is not enough to know “ x is vague” and “ y is

crisp”, to fully capture the meaning of the predicate “vague”. Actually, this is what, in fuzzy logic, is done by considering the predicate ‘fuzzy’ instead of ‘vague’, the mother-predicates of, respectively, the concepts of fuzziness and vagueness. Of ‘fuzzy’ it is easily possible to know when “P is less fuzzy than Q” in X, just if predicates P, Q, etc., can be represented in X by fuzzy sets with membership functions O_P, O_Q , etc., and by taking the so-called sharpened partial order between such fuzzy sets (De Luca, Termini, 1972).

c) A nice example lies in considering the differences between “odd” if used in the set N of natural numbers, or if used in a set of people. In the first case, that n in N is odd is just an “if and only if” definition, and the ordering of the statements “n is odd” is reduced to, for instance, “3 is equally odd than 107”. In the second case, it is neither possible to give such an “if and only if” definition for “John Doe is odd”, nor is it immediately clear how to order this type of elemental statements, besides it does not seem impossible.

2.2. The comparison among them of the P-elemental, or P-atomic, statements “x is P”, seems to be essential for obtaining some fruitful knowledge on the action, or behavior, of P in X. This comparison can be expressed by an algebraic relation that has, so often, some of the typical properties an algebraic order enjoys, and shows an elemental way on how P varies in X.

Notice that the used word “order” does not necessarily refer to a concept like the order between real numbers and that, in the discourse, can appear pairs of objects without links between them. For instance, it is possible that it is neither “John is less odd than Peter”, nor “Peter is less odd than John”, as it happens with complex number where it is neither “ $2-3i$ is less than $-5 + 2i$ ”, nor reciprocally. In fact, order refers here to some association of the pairs of statements in which some pairs can remain not associated, isolated.

3. The primary meaning

3.1. Let P be a predicate on a set, or universe of discourse, X, containing the “objects” that are submitted to a discourse. To primarily capture the meaning of P in X it seems necessary to know if, given x and y in X, it is, or it is not, “x is less P than y”, that is, if x shows p less than y shows it. Once this qualitative, empirical and perceptive relationship can be described, that is, once the behavior or use of P in X can be linguistically described, two algebraic relations \leq_p and \geq_p , both subsets of the Cartesian product $X \times X$, can be defined (Trillas, 2008) by,

$(x, y) \in \leq_p \text{ ó } x \leq_p y \text{ ó } x \text{ is P-less than } y \text{ ó } y \text{ is P-more than } x \text{ ó } y \geq_p x \text{ ó } (y, x) \in \geq_p$

showing that it is $\geq_p = \leq_p^{-1}$; that both relations are inverse to each other. Provided the relation \leq_p is not empty, it can be said that P is **meaningful** in X.

Of course, and provided the intersection of both relations is not empty, the relation

$$=_p = \leq_p \cap \geq_p,$$

reflects the empirical and perceptive relationship ‘P-equal than’, of equality in showing p.

Obviously, the non-emptiness of the relation $=_p$ is guaranteed if \leq_p enjoys the reflexive property, and $=_p$ is an equivalence relation if and only if \leq_p is a preorder, that is, it enjoys the reflexive and transitive properties. In this case, it exists the quotient-set $X/=_p$ consisting of the equivalence’s classes $[x] = \{y \in X; y =_p x\}$. This quotient-set is a partially ordered set once embedded with the partial order⁶ defined by

$$[x] \leq_p^* [y] \iff x \leq_p y,$$

that is not depending on the representatives chosen in both classes.

The algebraic relational structure, or graph, (X, \leq_p) , is the **primary meaning** of P in X, just reflects what is qualitatively known on the use of P in X, and is a first step that can conduct to measure its ‘extent’. Each qualitative use will be represented by a, perhaps different, relation \leq_p .

In the hypothesis that \leq_p is a preorder, it can analogously be taken $(X/=_p, \leq_p^*)$ as the primary meaning.

Example

With $X = [0, 10]$, and $P = \text{small}$, since “x is smaller than y” coincides with $x \leq y$ in the ordering of the real line, it can be established that the primary meaning of “smaller” in $[0, 10]$ given by $\leq_p = \leq$, is the graph $([0, 10], \leq)$. Since now \leq_p is reflexive and transitive, $=_p$ is an equivalence, it is $[x] = \{x\}$, and the two graphs (X, \leq_p) and $(X/=_p, \leq_p^*)$ are isomorphic.

As the predicate “big” is the opposite or antonym of “small”, it is $\leq_{\text{big}} = \leq_{\text{small}}^{-1} = \leq^{-1} = \geq$, the usual inverted order of the real line. Hence, the primary meaning of “big” in the interval $[0, 10]$ is given by the graph $([0, 10], \geq)$.

3.2. Notwithstanding, the primary meaning does not exhaust all the ways in which P can be used in X, since it only expresses how P grows in X, but not how much it grows. For doing such a typically scientific jump, more

⁶ In a partial ordered set (for short “poset”) (X, \leq_p) . not every pair of elements in X need be related by \leq_p .

information on the purpose on which P is used in X is required. Indeed, “use” cannot be separated from its “purpose”.

Remark

If it is $\leq_p = \emptyset$, P is **meaningless** in X , and if there is no way of even imagining how some relation \leq_p could be described, P is **metaphysical** in X . Hence, given a universe of discourse X , predicates that label properties of the elements in X are meaningful if $\leq_p \neq \emptyset$, meaningless if $\leq_p = \emptyset$, or metaphysical if they are neither meaningful, nor meaningless. Anyway, the border separating meaningless from metaphysical is certainly thin.⁷

In the set N of natural numbers the predicate “heavy” is, in principle, meaningless, unless it is given a new and specific definition or description of what it is understood by “ n is heavy” like, for instance, if the sum of the exponents in the decomposition in prime factors of n is greater than 1000. Notwithstanding, if “heavy” is certainly measurable in a universe X consisting of physical macroscopic objects, it also could be said to be measurable in, for instance, a universe of mathematical propositions provided a description of both “this theorem is a heavy one”, and “this theorem is less heavy than that” can be stated.

3.3. The character of being measurable or not being measurable should be always referred to a given use in a given universe of discourse; a predicate alone, without a universe is, at least, something a little bit bizarre.

⁷ In his *Tractatus* Wittgenstein distinguished between meaningful (sinnvoll), meaningless (sinnlos) and nonsensical (unsinnig) but he did not use the concept of metaphysical. The propositions of logic, i.e. tautologies and contradictions are senseless (sinnlos): “Tautologies and contradictions are not pictures of reality. They do not represent any possible situations. For the former admit all possible situations, and latter none.” (Wittgenstein, 1922, 4.462). By contrast, senseful propositions may “they range within the truth-conditions drawn by the propositions of logic. But the propositions of logic themselves are neither true nor false.” [...]

Other (non-logical) propositions can be senseless because they may apply to things that “cannot be represented, such as mathematics or the pictorial form itself of the pictures that do represent. These are, like tautologies and contradictions, literally sense-less, they have no sense.” [...]

Nonsensical (unsinnige) propositions cannot carry sense, e.g. “Socrates is identical”. “While some nonsensical propositions are blatantly so, others seem to be meaningful—and only analysis carried out in accordance with the picture theory can expose their nonsensicality. Since only what is “in” the world can be described, anything that is “higher” is excluded, including the notion of limit and the limit points themselves. Traditional metaphysics, and the propositions of ethics and aesthetics, which try to capture the world as a whole, are also excluded, as is the truth in solipsism, the very notion of a subject, for it is also not “in” the world but at its limit.” (Biletzki, Matar, 2011).

Predicates enter in the language by being applied to particular objects; for instance, possibly “heavy” begun by being used with rocks, latter on passed to be applied to pieces of wood, pieces of iron, ... , and so on. It is for this reason that for a lot of words, dictionaries should show several possible practical applications for using them. Of course, the study of the evolution of predicates in language is an interesting subject (García-Honrado, Trillas, 2011) still waiting for a complete enough mathematical treatment.

Nevertheless, and in the set of all currently existing living beings, the predicate “eternal”, provided it were applicable, does not allow to state that any living being is eternal, and less again that ones are less, or more, eternal than others; hence “eternal” is metaphysical if applied in this universe of discourse. A different question arises in questioning if there exists some universe in which eternal is not metaphysical; a predicate can be even called as “**now it is metaphysical**” when no universe in which it could be, at least in principle, measurable is currently known. Anyway, it is neither at all the task of this paper, nor the wish of their authors, to consider those predicates that currently are metaphysical.

Like a meaningless predicate can be transformed in a measurable one by a suitable new description of how it is used, also metaphysical predicates can loss such character provided the predicate could be correctly and newly described in the context it appears. This is a joint matter concerning both the existing ground for its use (be it real or virtual) and the linguistic analysis of what is said; anyway, clarifying the character of the involved predicates is always a sane attitude towards a good enough understanding on anything and, many times, the use of analogies can hide such character. Although the use of metaphorical reasoning is well acceptable in some occasions, it should always be submitted to a careful control ^[15]. What is sure is that only meaningful predicates can be studied, or specified, through a scientific-like process. In this sense this paper could be inscribed under the label ‘positivistic’, in a similar way that the members of the Vienna Circle accepted such qualification.

4. Meanings

4.1. Non meaningless, and non metaphysical predicates, that is meaningful predicates, those whose primary meaning is given by a graph (X, \leq_p) with $\leq_p \neq \emptyset$, are **measurable** predicates; they can be considered from a scientific point of view allowing to measure them.

To such an end let us define that a measure in a graph (X, \leq_p) is a mapping $t_p: X \mapsto \mathbf{R}^+$, such that:

- 1) If $x \leq_p y$, then $t_p(x) \leq t_p(y)$, in the order of \mathbf{R}^+ ,
- 2) If x is a minimal for \leq_p , then $t_p(x) = \text{Inf } t_p(X)$,
- 3) If x is a maximal for \leq_p , then $t_p(x) = \text{Sup } t_p(X)$.

The triplet (X, \leq_p, t_p) is but a numerical quantity, and t_p tries to reflect up to which extent each x is P , or verifies p . That is, $t_p(x)$ reflects the “amount of p ” carried by each $x \in X$.

Notes

a) Property 1 is essential. Provided t_p were not constantly non-decreasing when the elements in X , “increase” under \leq_p , a mapping t_p cannot correctly reflect how varies the extent of P in X . In addition, it will be for nothing.

b) x is minimal provided there is no z in X such that $z \leq_p x$, and y is maximal if there is no z in X such that $y \leq_p z$. When there exists a single minimal it is called the minimum, and when there is a single maximal is the maximum.

c) Like it happens so often with formal definitions, as it is, for instance, with that of a probability, “axioms” 1 to 3 are insufficient to specify a single measure, and some additional information, either on \leq_p , or in the shape of t_p , etc., is necessary for determining a particular measure. Once one such a measure is specified, the corresponding quantity is itself but a **model** for the meaning of P in X or, for short, a model of P .

d) When $t_p(X)$ is not a bounded subset of \mathbf{R}^+ , then it is $\text{Sup } t_p(X) = +\infty$, or $\text{Inf } t_p(X) = -\infty$, but, in what follows it will be supposed that it is $t_p(X) = [0, 1]$. Provided $t_p(X)$ were an interval $[a, b]$, it will suffice to take the linear transformation

$$t_p^*(x) = (a - t_p(x))/(a - b),$$

preserving (1), for having the values of the measure in the unit interval. Since in such case it is $\text{Inf } t_p(X) = a$, and $\text{Sup } t_p(X) = b$, the extreme values of t_p^* will be 0 and 1.

e) In his book on Calculus ^[16], Karl Menger views quantities in a close form to what here is the family of pairs $\{(x \text{ is } P, t_p(x)); x \in X\}$, and it could be interesting to quote the statement in which he says that, the “laws connecting quantities are found by observation and experiment -not by logic or mathematics-“.

This quotation just recalls that the quantities (X, \leq_p, t_p) try to approach the concept of meaning by grounding it in a common language’s use experience. Something that is simply done with the typically Vienna Circle’s

‘delight for the explication of concepts’, as Bert Schweizer and Abe Sklar said in their Preface to Menger’s book re-edition.

Example

In the case of “small” in $[0, 10]$, the measures t_{small} are the functions $t_{\text{small}} : [0, 10] \mapsto [0, 1]$, such that

$$1^*) \text{ If } x \leq y, \text{ then } t_{\text{small}}(x) \leq t_{\text{small}}(y)$$

$$2^*) t_{\text{small}}(0) = 1$$

$$3^*) t_{\text{small}}(10) = 0.$$

Hence the measures t_{small} are the numerical functions decreasing from the point $(0, 1)$ to the point $(10, 0)$ in the region $[0, 10] \times [0, 1]$ of the Euclidean plane and, of course, there is an infinite number of such functions. To specify just one, more information is required; an information that could come from knowing more on the qualitative behavior of “small”, or from a reasonable hypothesis on the shape of t_{small} , made to continue its study. For instance, if such information or hypothesis can conduct to suppose that the measure should be linear, then the only suitable measure is $t_{\text{small}}^{(1)}(x) = 1 - x/10$, and the meaning itself is just the only linear model for “small”. Nevertheless, and provided some information or hypothesis conducts towards a quadratic measure, there are many functions $t_{\text{small}}(x)$ of the form $ax^2 + bx + c$, and verifying (1^*) to (3^*) . One of them is $t_{\text{small}}^{(2)}(x) = (1 - x/10)^2 = x^2/100 - x/5 + 1$, that can be called a quadratic model of “small”. Notice that $t_{\text{small}}^{(1)}(5) = 0.5$, but $t_{\text{small}}^{(2)}(5) = 0.25$.

Additionally, if it can be hypothesized that the measure should be piecewise-linear and with $t_{\text{small}}^{(3)}(5) = 0.4$, it will result,

$$t_{\text{small}}^{(3)}(x) = 1 - 3x/25, \text{ if } 0 \leq x \leq 5,$$

$$t_{\text{small}}^{(3)}(x) = (4 - 0.4x)/5, \text{ if } 5 \leq x \leq 10,$$

and the meaning itself can be called a piecewise-linear model.

Provided the only available information were “nothing is known on the smallness of x in the open interval $(0, 10)$ ”, then, and by applying the Bernoulli-Laplace’s principle of insufficient reason⁸, it could be taken: $t_{\text{small}}(0) = 1$, $t_{\text{small}}(10) = 0$, and $t_{\text{small}}(x) = 0.5$ for all x in $(0, 10)$. This is a perhaps

⁸ If we are ignorant of the ways an event can occur (and therefore have no reason to believe that one way will occur preferentially compared to another), the event will occur equally likely in any way. This principle was considered first and intuitively obvious by Jakob Bernoulli and Pierre Simon Laplace. Later, the principle was given the name “principle of insufficient reason” by George Boole and John Venn in contrast to the principle of sufficient reason of Leibniz. Later, Poincaré (Poincaré, 1912) and (Keynes, 1921) emphasized this principle in their books on probability theory.

insufficient measure just reflecting ignorance on the behavior of “small” in $(0, 1)$, and by supposing 0.5 the mid-point between ‘absolutely yes’ (1) and ‘absolutely not’ (0), that depends on the working hypothesis that $t_{\text{not small}}(x) = 1 - t_{\text{small}}(x)$. For instance, provided the negation were ‘not linear but distorted’, as it happens with the Sugeno’s negation-function⁹, $t_{\text{not small}}(x) = [1 - t_{\text{small}}(x)] / [1 + t_{\text{small}}(x)]$, then $t_{\text{small}}(x) = 0.5$ would imply $t_{\text{not small}}(x) = 1/3 \neq 0.5$.

4.2. Consequently, for a fixed primary meaning of P in X, there is not a single quantity (X, \leq_p, t_p) giving the numerical extent up to which the elements x in X carry P. In general, there are many of such quantities each one reflecting a possible use of P in X, according to a particular purpose for such use and from which additional information arises. Each quantity (X, \leq_p, t_p) will be called a **meaning** of P in X. In principle, as shown by the example given by “odd”, the meaning of P depends on the universe, the context and the purpose for its use.

Sometimes the existence of a unique model of the meaning, like it is the former case with t_{small} being a linear function, can suggest to adopt this model up to when new information could advise to change it.

Provided \leq_p is a preorder, it can be defined the function $t_p^*: X/\equiv_p \rightarrow [0, 1]$, $y t_p^*([x]) = t_p(x)$, since it is obviously $x \equiv_p y \Rightarrow t_p(x) = t_p(y)$. Of course, t_p^* is a measure and, in this case, also the quantity $(X/\equiv_p, \leq_p^*, t_p^*)$ can be taken as the meaning of P in X.

5. Working meanings

Once a model quantity is known for describing a meaning, a new algebraic relation \leq_{ip} is automatically defined by,

$$x \leq_{ip} y \Rightarrow t_p(x) \leq t_p(y).$$

This new relation verifies:

$$x \leq_p y \Rightarrow t_p(x) \leq t_p(y) \text{ ó } x \leq_{ip} y,$$

showing that it is,

$$\leq_p \subseteq \leq_{ip},$$

the new relation is larger than that giving the primary meaning. With each measure of the primary meaning some information is added to it, the primary meaning is extended, and to have $\leq_p = \leq_{ip}$ it is necessary that the first relation is linear, since the second is so because for all x, y in X it will be always either $t_p(x) \leq t_p(y)$, or $t_p(x) > t_p(y)$. If all pair of elements x

⁹ See (Sugeno, 1977).

and y , are always comparable under the relation \leq_{t_p} , this is not always the case under \leq_p . For instance, and under the three before mentioned measures $t_{small}^{(i)}$, there is identity between both relations in $[0, 10]$, but there is no coincidence with the predicate $Q = \text{around four}$, in the same interval, for which it is

$$x \leq_Q y \Rightarrow \text{either } 0 \leq x \leq y \leq 4, \text{ or } 4 \leq y \leq x \leq 10,$$

a relation that is not linear. For instance, it is neither $2 \leq_Q 7$, nor $7 \leq_Q 2$; 2 and 7 are not comparable under \leq_Q . Hence, \leq_Q is strictly contained but different to \leq_{t_Q} , for any measure t_Q .

The pair (X, \leq_{t_p}) will be called a **working-meaning** of P in X , since usually the working scientist only works with a measure and ignores the primary meaning. When $\leq_p = \leq_{t_p}$ it will be said that the measure t_p **perfectly reflects** P 's primary meaning; that meaning and working-meaning coincide.

Of course, the relation \leq_{t_p} is total or linear just because the natural ordering of the real line is total. Provided the values of the measures were in a partially ordered set, like it is that of the complex numbers, more chances for the coincidence of primary and working meanings could be open, even if no guarantee on it can be offered. In general, measuring alters the qualitative or primary meaning.

Remark

The case with probability is a good enough one to see that what is written before is not at all rare.

Provided the predicate $P = \text{probable}$ is applied to a Boolean algebra Ω of subsets of a universe X , and also provided the relation \leq_{probable} coincides with the inclusion (\subseteq) between the subsets of X , all probabilities $p : \Omega \rightarrow [0, 1]$ are measures of P . Indeed, by definition it is $p(\emptyset) = 0$, $p(X) = 1$, and if $A \subseteq B$, from $p(B) = p(A \cup (B \cap A^c)) = p(A) + p(B \cap A^c) \geq p(A)$, due to the additive law of p . Each quantity (Ω, \subseteq, p) reflects a use, or meaning, of P in Ω and also, and in particular in X , whenever it will be possible to identify the singletons $\{x\}$ that could belong to Ω , with the elements x in X . Obviously, there can exist subsets A and B in Ω , not comparable by inclusion, but verifying $p(A) \leq p(B)$, or $A \leq_p B$; in most cases probabilities cannot perfectly reflect the primary meaning (Ω, \subseteq) of P .

This example allows seeing that the measure depends on either the information available on the problem, or on the hypotheses supplied by the designer. For instance, the usual and unrealistic hypothesis that a dice is perfect, or ideal, conducts to take the working probability p defined by $p(\text{obtaining } 1) = \dots = p(\text{obtaining } 6) = 1/6$. Provided it were known that the dice is trickily loaded in such a way that $p(\text{obtain } 6) = p(6) = 0.4$, then all

probabilities verifying: $p(1) + p(2) + p(3) + p(4) + p(5) = 1 - p(6) = 0.6$, are admissible, although either additional information, or a hypothesis like the ‘uniform’ one $p(1) = p(2) = p(3) = p(4) = p(5) = 0.6/5 = 0.12$, is necessary to fix a working probability.

6. Complex measures

There are some real situations in which the values of the measure cannot be exactly calculated as a real number, but the only that can be stated is that $t_p(x)$ belongs to some numerical sub-interval $[a(x), b(x)]$ of the unit interval $[0, 1]$. In such cases, there is the possibility of changing the real unit interval by the complex one,

$$U = \{a + ib; a, b \text{ in } [0, 1]\},$$

that, with the partial order

$$a + ib \leq a^* + ib^* \Rightarrow a \leq a^* \text{ and } b \leq b^*,$$

of the ordered field of complex numbers, is just a partially ordered set with minimum $0 = 0 + i0$, and maximum $1 = 1 + i$. Thus, complex measures will be the mappings $t^*: X \rightarrow U$, such that,

$$1^*) \ x \leq_p y \Rightarrow t^*(x) \leq t^*(y)$$

$$2^*) \ \text{If } x \text{ is minimal under } \leq_p, \text{ then } t^*(x) = 0$$

$$3^*) \ \text{If } y \text{ is maximal under } \leq_p, \text{ then } t^*(y) = 1,$$

and for all x in X , it will be $t^*(x) = t_r(x) + it_i(x)$ in U .

Hence, the working-meaning will be given by

$$x \leq_{t^*} y \Rightarrow t^*(x) \leq t^*(y),$$

a usually non-total relation that could result suitable in cases where \leq_p is not linear. Of course, and when \leq_p is not enjoying a total, or linear, character¹⁰, it is not sure that t^* will allow to perfectly reflect the primary meaning, but obviously some additional possibility appears in such case.

It should be pointed out that complex measures are not at all something rare in science and, even if their values are represented in the modulo-argument form, $t^*(x) = r_A = r \cdot (\cos A + i \cdot \sin A)$, it can additionally facilitate some geometrical translation of the current problem collecting new and perhaps fruitful interpretations, like it happens in Electrodynamics.

Returning to when the most that can be said of the values $t(x)$ is that they belong to closed intervals $[t_1(x), t_2(x)]$, it is possible to define a complex measure $t^*(x) = t_1(x) + it_2(x)$ like it is, for instance,

¹⁰ A total ordered set (X, \leq_p) is a poset with the additional condition (“comparability condition” or “trichotomy law”: For all x, y in X it is either $x \leq_p y$ or $y \leq_p x$).

$t^*(x) = t_2(x) + i((t_2(x) - t_1(x))/2)$,
 that verifies properties (1*), (2*), and (3*), and that simply presupposes the ordering

$[a, b] \leq [a^*, b^*] \Rightarrow a \leq a^* \ \& \ b \leq b^* \Rightarrow a+ib \leq a^*+ib^*$,
 between intervals, that is that $[a, b]$ is less something than $[a^*, b^*]$ if the first is less at right than the second. What should be taken into account is nothing else than the set of intervals is order-isomorphic with U .

7. Meaning and fuzzy sets

A fuzzy set P , labeled P in a set X , is represented by means of its membership function (Zadeh, 1965) $\mathcal{O}_p: X \rightarrow [0, 1]$, reflecting that, for all x in X , the number $\mathcal{O}_p(x) \in [0, 1]$ is the “grade up to which x is P ”.

Fuzzy sets theorists or practitioners avoid, even declaring that with \mathcal{O}_p they try to capture the meaning of the linguistic term P in X , not only to which meaning’s interpretation they refer to, but how the membership function behaves with respect to it, or what they mean by “grade”; in short, they avoid seeing fuzzy sets as the measures t_p introduced in section 4, that allow to see fuzzy sets as informational-states of the linguistic label (Trillas, 2008). This, jointly with some insistence that fuzzy logic should be seen through the glasses of multiple-valued logic, provoked a very curious mixture of viewings on fuzzy “logic”. Under them and, for instance, if engineers always represent the linguistic rules of the type “If x is P and y is Q , then z is R ”, describing the behavior of a dynamic system, by a (commutative!) intersection of the fuzzy sets representing the rule’s antecedent and consequent, there are in the field some theoretically-driven mathematicians insisting that only some generalizations of the classical material implication, $\mathcal{O}A \mathcal{U} B$ - not the antecedent or the consequent - a non commutative operation, should be used for this goal, and even if the negation of the antecedent is not a describable fact. There is a remarkable disparity in the affording of problems between those who are practice-driven and those who are theory-driven; something that being not exceptional between professional practitioners and academicians seems to be new in a modern research field.

Such differences are, in the authors’ view, due to an opposition between “practical” and “logical” approaches to fuzzy logic’s true problems, and concerning the mathematical representation of linguistic statements. If the practical one could be reinforced by its applicative success, the logical one just lies in the believe that logic (in the modern sense of the deductive one)

is the correct tool for analyzing and representing language and common sense reasoning (Trillas, 2008), and this paper would like to contribute to support a view under which language and ordinary reasoning, at the end natural phenomena, should be approached through some similar tools to those of experimental sciences, as Lord Kelvin's wrote:

I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely in your thoughts advanced to the state of Science , whatever the matter may be. (Thomson, 1883)

This text, often shortened as the strong dictum “If you cannot measure it, is not science”, can force to reflect on what is or is not measurable, and from where comes the possibility of introducing a measure for previously empirically reached concepts.

A first step towards such science-like approach, is to have accurate knowledge of the existing good practical problems and their best solutions, like it happens with intentional observation and controlled experimentation in natural sciences. If, although coming later on, this seems to be essential for the future fertility of such approach, of course, the today widely accepted name “fuzzy logic” does not contribute to it. Perhaps, either the short name “fuzzy science”, or the larger “science of imprecision and non-random uncertainty” (Trillas, 2013) could be more suitable in the future.

As it is well known by practitioners, and even if they not always explicitly recognize it, the membership function of a fuzzy set is not only non-unique, but each one should be carefully designed accordingly with the best information currently available on the behavior of the linguistic label in X , including its context and purpose (or hypothesis) for its use. Since many predicates appearing in language, like “beautiful” applied to works of art, are not easily translatable into a numerical scale, like it is the case for “tall” through the numerical height in centimeters of the involved people, it is sometimes very difficult to well recognize the full relation \leq_p and, hence, the only that can be said is that O_p is a sort of approximation to a measure t_p . Something similar to the universal approximation character existing between the real behavior of a dynamic system described by linguistic rules, and figured out by a curve or surface, and its approximation through the function giving the numerical outputs of Zadeh's Compositional Rule of Fuzzy Inference (Zadeh, 1973) once the rules and the inputs are represented by fuzzy sets, and a defuzzification method is selected. At this respect, a lot of work is still to be done.

Anyway, and either knowing it or not, researchers in fuzzy logic look at the predicates, or linguistic labels as they call them, from the Wittgenstein's point of view, stated in his 'Philosophical Investigations' by, "The meaning of a word is its use in language" (Wittgenstein, 1958).

8. Conclusions

8.1. Although basically this paper only deals with meaningful predicates and it does not consider metaphysical statements, but only tries to clarify what a metaphysical concept or predicate is, it is not at all in its goal to expulse metaphysical statements from the large world of reasoning. By the way, a statement can be defined to be metaphysical whenever some of its components do concern a metaphysical concept or predicate.

The only things on which the authors are convinced of is on the relevance, for the advancement of science, of those concepts that can be measured, but they also do recognize the possible value for the general advancement of knowledge of those that are not immediately measurable. Especially when what is in play is to reach "new" ideas, that is, ideas that cannot be deduced from the reasoning's premises but that are able to enlarge its information content. This type of non deductive reasoning is also absolutely crucial for the advancement of science, art, philosophy, history, etc., where new and fertile ideas and concepts actually guarantee their progress. A type of reasoning not well completely studied in the branch of knowledge known as "logic", itself limited to formally analyze the rigidly deductive and formal processes of reasoning once translated into an artificial language.

All that deserves here a short stop to quote the 'Law of Conservation of Information' established by the biologist and Nobel Laureate Sir Peter B. Medawar. He wrote (Medawar, 1984):

"No process of logical reasoning can enlarge the information content of the axioms and premises or observation statements from which it proceeds", implying that for studying the growing of the information content the clothes of current logic are too short, and even too large. It is not in the genetic resources and basic intellectual interests of the rigid modern deductive logic, a tendency to model all processes of reasoning and, in particular, of those that to be creative should proceed throughout a reasoning that is intrinsically speculative, neither always consistent, nor monotonic." (Trillas, 2012). These are the processes of reasoning searching for new fertile ideas, those creatively conducting to actually enlarge the information content of the previous knowledge, and where some non-

meaningful but metaphysical concepts can play an important, even crucial, role by guiding the thinker towards something new and thanks to its open meaning; an open meaning allowing the thinker - be it an artist, a novelist, a scientist, a philosopher, a historian, a physician, etc. - to more or less slowly advance between the fog of what is yet unknown. This is a subject still waiting for, and well deserving, more elaboration.

8.2. It should be pointed out that the definition of a real or complex measure only depends on the perceived primary meaning \leq_p and on the order structure of the real or complex unit interval, but not on the nature of the elements in such interval. Hence, the unit interval can be easily substituted by a partial ordered set (L, \leq) , with minimum 0 and maximum 1, and on L-measures that can be defined as the mappings $t: X \rightarrow L$, such that:

- 1) $x \leq_p y \Rightarrow t(x) \leq t(y)$;
- 2) If z is a minimal for \leq_p , then $t(z) = 0$;

and

- 3) If z is a maximal, then $t(z) = 1$.

In this way a window is open towards a qualitative, not necessarily numerical, concept of meaning. For instance, if the needed qualitative measuring is of the type “the measure of the extent up to which x is P , is high”, “the measure of the extent up to which y is P , is low”, etc., the partially ordered set of values can be obtained from the set L^* of linguistic predicates, $L^* = \{\text{low, middle, high, ...}\}$, once they are represented by the fuzzy sets (Trillas, 2013) in $L^{**} = \{A_{\text{low}}, A_{\text{middle}}, A_{\text{high}}, \dots\}$ that, ordered pointwise: “ $A \leq B$ if and only if $A(x) \leq B(x)$, for all x in X ”, typical of fuzzy sets, and completed if necessary with the extremes A_0 and A_1 , gives the set $L = L^{**} \cup \{A_0, A_1\}$ and, finally, the poset (L, \leq) . Recall that the fuzzy sets A_0 and A_1 , are respectively defined by $A_0(x) = 0$, and $A_1(x) = 1$, for all x in X . Thus, and for instance, $t(x) = A_{\text{middle}}$ just translates the linguistic statement “the measure up to which x is P , is middle”, or equivalently, “the measure of the amount of P carried by x is middle”.

If \leq_p is a preorder, then with the partially ordered set $(L, \leq) = (X/\equiv_p, \leq_p^*)$, we can define the qualitative measure $t^*(x) = [x]$, assigning to each x in X the equivalence class it represents and that, perhaps, only has a theoretical interest. Since,

$$x \leq_p y \Rightarrow [x] \leq_p^* [y] \Rightarrow t^*(x) \leq_p^* t^*(y) \Rightarrow x \leq_{tp} y,$$

it immediately follows $\leq_p = \leq_{tp}$, that is, the L - working meaning coincides with the primary meaning and, hence, that t^* perfectly reflects the primary meaning.

8.3. When the American Association for the Advancement of Science organized in 1966 a symposium to commemorate the 50th anniversary of Ernst Mach's death, Karl Menger participated with a contribution on Positivistic Geometry (Menger, 1970) that he subtitled by "A Probabilistic Microgeometry". He saw the most difficult problem for establishing such microgeometry lies in the individual identification of elements of the space. Already in 1951, when he was guest resident at the Sorbonne, he had published a suggestion for this problem where in addition to studies of well-defined sets, he called for a theory to be developed in which the relationship between elements and sets is replaced by the probability that an element belongs to a set:

Une relation monaire au sens classique est un sous-ensemble F de l'univers. Au sens probabiliste, c'est une fonction Π_F définie pour tout $x \in U$. Nous appellerons cette fonction même un ensemble flou et nous interpréterons $\Pi_{F(x)}$ comme la probabilité que x appartienne à cet ensemble. (Menger, 1951a)

In English-written papers, Menger later replaced the term "ensemble flou" with the expression "hazy set". Hazy sets can be seen as a probabilistic antecedent of Zadeh's fuzzy sets.

In one of those papers, (Menger, 1951b) also published in 1951, Menger defined, in a set S , a "statistical metric" if a probability function $\Pi(x; p, q)$ that is linked with two of its elements p and q satisfies the following conditions

1. $\Pi(0; p, p) = 1$, (The probability is 1 that the distance between p and p is 0.)
2. If $p \neq q$, then $\Pi(0; p, q) < 1$,
3. $\Pi(x; p, q) = \Pi(x; q, p)$,
4. $T[\Pi(x; p, q), \Pi(y; q, r)] \leq \Pi(x+y; p, r)$, (Triangle inequality)
where $T[\alpha, \beta]$ is a function defined for $0 \leq \alpha, \beta \leq 1$ such that:
 - (a) $0 \leq T[\alpha, \beta] \leq 1$.
 - (b) T is non-decreasing in every variable.
 - (c) $T[\alpha, \beta] = T[\beta, \alpha]$.
 - (d) $T[1, 1] = 1$.
 - (e) If $\alpha > 0$ then $T[\alpha, 1] > 0$

Menger called $\Pi(x; p, q)$ the distance function of p and q and interpreted it as the probability that the distance between points p and q is $\leq x$. The "triangle inequality" implies that the following is true for all points q and all numbers x between 0 and z :

$$\Pi(z; p, r) \geq \text{Max } T[\Pi(x; p, q), \Pi(z - x; q, r)].$$

He called T a “triangular norm” (“t-norm”). Only later on, and to extend the triangular inequality to more than three points, Menger’s t-norms were supposed associative. Associative and continuous t-norms are perfectly classified and extensively used in fuzzy logic.

In another paper he considered then the probability $E(a, b)$ that points a and b in a universe of discourse X are indistinguishable, that has the following properties:¹¹

- (1) $E(a, a) = 1$ for all a in X ;
- (2) $E(a, b) = E(b, a)$, for all a and b in X ;
- (3) $E(a, b) \cdot E(b, c) \leq E(a, c)$, for all a, b, c in X .

He proposed calling ‘ a and b certainly-indistinguishable’ if $E(a, b) = 1$. This led to an equality relation, since all elements that are certainly-equal to a can be combined into an equality set A , and each pair of these sets is either disjoint or identical.

He defined $E(A, B)$ as the probability that every element of A is equal to every element of B . This number is not dependent upon the selection of the two elements in each case.

In his paper of 1966 Menger intended to use the concept of hazy sets in microgeometry, but he clearly recognized that it was simply not possible to identify the individual elements of hazy sets which were after all defined by means of probabilities. Though, he suggested combining the concept of hazy sets with a geometry of “lumps”, for lumps were easier to identify and to differentiate from one another than points. Lumps could assume a position between indistinguishability and apartness, which would be the condition of overlapping. It was irrelevant whether the primitive (i. e. undefined) concepts of a theory were characterized as points and probably indistinguishable, or as lumps and probably overlapping. Of course, all of this depended on the conditions that these simple concepts had to fulfill, but the properties stipulated in the two cases could not be identical.

I believe that the ultimate solution of problems of microgeometry may well lie in a probabilistic theory of hazy lumps. The essential feature of this theory would be that lumps would not be point sets; nor would they reflect circumscribed figures such as ellipsoids. They would rather be in mutual probabilistic relations of overlapping and apartness, from which a metric would have to be developed. (Menger, 1970, 233)

In the here presented view on meaningful predicates as measurable predicates, given by a graph (X, \leq_{P-CE}) , where P is the predicate for the –

¹¹ Menger commented that conditions (1) and (2) correspond to reflexivity and symmetry for the equality relation, while condition (3) “expressed a minimum of transitivity” (Menger, 1951b).

microgeometrical property $p = \text{location}$, we can identify the elements of X with 1) points that are probably indistinguishable or with 2) lumps that are probably overlapping. In both cases $(X, \leq_{p=CE})$ is a poset. In the first case the relation $\leq_{p=CE}$ is the certainly-indistinguishable-relation $E(a, b)$ of points in X , in the second case $\leq_{p=CE}$ is the certainly-indistinguishable-relation $E(A, B)$ of equally sets.

All that was introduced by Menger under his view that the difficulties on the intuition of what quantum physics establishes, lies in the lack of a suitable geometry of the microcosmos. The paper he wrote for the book 'Albert Einstein, scientist and philosopher', in the Library of Living Philosophers (Menger, 1949), shows an interesting overview on how evolved geometry in parallel with the world's knowledge.

8.4. Although the authors of this paper do not fully share all that positivistic philosophy tried and meant, they must recognize that for the current manifestations of human thought, strongly influenced by the views of scientific practice, it is not a superfluous exercise that of trying to look at old philosophical problems from some more or less modern scientific perspectives. On the contrary, philosophy will be mainly restricted to other, even relevant, affairs.

Of course, this implies that philosophically-oriented thinkers should not only know the history of science, but also they should have a good comprehension of what scientists technically did, and especially for what and how they did it. Of course, this implies that scientifically-oriented thinkers should try to capture that with which philosophers contributed to the philosophy of knowledge.

Philosophy is not a land closed by crisp boundaries, but an open land for reflection where, like elsewhere, both analogy and exact reasoning play a paramount role. This paper, convinced as the authors are on the convenience of keeping a deep interrelation between different ways of looking at knowledge, tries to favor a joint endeavor between both philosophically and scientifically oriented thinkers; and for which it is necessary an effort to approach the different ways of thinking, on what are the worries in the back of them, and on the languages they are expressed. Something that is not actually easy.

Acknowledgment

This paper is partially funded by the Foundation for the Advancement of Soft Computing, and by the Government of Spain project MICIIN/TIN 2011-29827-C02-01.

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